

Asymptotic symmetries in an optical lattice

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It was recently remarked by Lutz [*Phys. Rev. A* **67** (2003), 051402(R)] that the equation for the marginal Wigner distribution in an optical lattice admits a scale-free distribution corresponding to Tsallis statistics. Here we show that this distribution is invariant under an asymptotic symmetry of the equation, hence that this scale-free behavior can be understood in terms of symmetry analysis.

In a recent paper with R. Mancinelli [1] we have considered asymptotic symmetries of differential equations and of their (asymptotic) solutions; in there we considered in particular application to a given class of anomalous reaction-diffusion equations which had been extensively studied numerically [2], obtaining a theoretical explanation of the observed long-time behavior of solutions. The same approach has also been extended to a discrete version of these equations [3].

In this note we apply our approach to a different kind of anomalous diffusion; that is, we focus on the equation describing anomalous transport in an optical lattice [4, 5]. The equation for the marginal Wigner distribution $w(p, t)$ of the momentum p at time t reads

$$w_t = -\frac{\partial}{\partial p} [h(p)w - g(p)w_p] \quad (1)$$

where the functions $h(p)$ and $g(p)$ are given by

$$h(p) := \frac{-\alpha p}{1 + (p/p_c)^2}, \quad g(p) := \gamma_0 + \frac{\gamma_1}{1 + (p/p_c)^2}; \quad (2)$$

here $\alpha, \gamma_0, \gamma_1$ are certain constants, p_c represents the capture momentum.

Equation (1) should be complemented by a choice of the function space to which the function $w(p, t)$ belongs. On physical grounds, we require that $w(p, t)$ at fixed t is normalizable and vanishes asymptotically,

$$|w(p, t)| \rightarrow 0 \quad \text{for } |p| \rightarrow \infty. \quad (3)$$

We stress that there is a range of parameters ($-1/2 < \mu$) [5] for which there is an equilibrium distribution with these properties; this is the range to be studied here.

The explicit expressions of $h(p)$ and $g(p)$ make that, with the shorthand notation $\beta(p) := 1/[1 + (p/p_c)^2]$, the equation (1), (2) reads

$$w_t = (\gamma_0 + \beta(p)\gamma_1) w_{pp} + (\alpha\beta(p) - 2\gamma_1[\beta(p)/p_c]^2) p w_p + \alpha (\beta(p) - 2[(p/p_c)\beta(p)]^2) w. \quad (4)$$

Attention to eq. (4) was recently called by Lutz [5] (to which the reader is also referred for derivation and a

discussion of this equation), who remarked that – quite surprisingly – the equilibrium distribution is a Tsallis distribution, i.e. a distribution optimizing the Tsallis entropy [6]. Indeed, the stationary solution of (4) turns out to be

$$w_0(p) = \frac{1}{Z} [1 - (\beta/\mu)p^2]^\mu \quad (5)$$

where

$$\beta = \frac{\alpha}{2(\gamma_0 + \gamma_1)}, \quad \mu = \frac{1}{1 - q}, \quad q = 1 + \frac{2\gamma_0}{\alpha p_c^2} := 1 + \delta, \quad (6)$$

and Z is a normalization factor, which can be chosen so that $\int_{-\infty}^{\infty} w_0(p) dp = 1$. The physical range is $q < 3$; the case $5/3 < q < 3$ (for which the second moment $\int p^2 w_0(p) dp$ diverges) corresponds to anomalous moment diffusion [5].

The peculiar properties of this equation and its stationary solution, in particular concerning the power-law decay of $w_0(p)$ for large $|p|$, hence its scale-free nature, has been studied by Abe [7] in connection with dilation symmetries and canonical formalism.

Here we will consider generalized scaling transformations; that is, local transformations acting as a standard scaling in the independent p and t variables, and as a p and t dependent scaling in the dependent variable $w(p, t)$. Moreover, we will not require to have a full invariance of the equation, but be satisfied in *asymptotic invariance* (for large $|p|$ and t and hence, in view of (3), small w), as discussed in detail in [1].

We start by recalling a general feature of symmetry transformations and symmetry invariance for PDEs and their solutions [8–10]; we will specialize to a scalar PDEs for $w = w(p, t)$.

Under an infinitesimal transformation with generator

$$X = \xi(p, t, w)\partial_p + \tau(p, t, w)\partial_t + \varphi(p, t, w)\partial_w \quad (7)$$

(that is, $p \mapsto \hat{p} = p + \varepsilon\xi(p, t, w)$, and so on) the function $w(p, t)$ is mapped to a new function $\hat{w}(p, t)$ with $\hat{w}(p, t) = w(p, t) + \varepsilon[\delta w(p, t)]$ and

$$\delta w = [\varphi - w_p\xi - w_t\tau]_{w=w(p, t)}. \quad (8)$$

Applying this to the distribution (5) and to generalized scalings we get with easy computations that (up to a common factor) the transformation generated by (7) with $\xi = -p$ leaves (5) invariant if and only if

$$\varphi = (2\beta Z^{q-1})p^2 w^q; \quad (9)$$

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obviously $\tau = \sigma t$ remains unrestricted at this stage. Conversely, a function $w(p)$ is invariant under $X = -p\partial_p + (2\beta Z^{q-1})p^2 w^q \partial_w$ if and only if $w(p)$ is of the form (5); this is easily checked by using (8). We will write, from now on,

$$\delta = q - 1, \quad \nu = 2\beta Z^\delta. \quad (10)$$

Albeit our analysis will be conducted at the infinitesimal level, we note that the action of the one-parameter group (denote by s the group parameter) generated by $X = -p\partial_p + \nu p^2 w^{1+\delta} \partial_w$ on the p and w variables is to map $(p, w) = (p_0, w_0)$ into $(p(s), w(s))$ with

$$\begin{aligned} p(s) &= e^{-s} p_0, \\ w(s) &= [1 + (\nu p_0^2/2)(1 - e^{-2s})w_0^\delta]^{-(1/\delta)} w_0. \end{aligned} \quad (11)$$

We are now going to discuss what is special in the generalized scaling vector fields identified above

$$X = -p\partial_p + \sigma t\partial_t + (\nu p^2 w^q) \partial_w \quad (12)$$

for what concerns their action on the equation (4).

We follow the general procedure for group analysis of differential equations [8–10]. By acting with X on the (p, t, w) variables, we induce a transformation in the partial derivatives of w with respect to p and t as well (also called the second prolongation of X); this is described – restricting to partial derivatives occurring in (4) – by the vector field

$$Y = X + \Psi_t \frac{\partial}{\partial w_t} + \Psi_p \frac{\partial}{\partial w_p} + \Psi_{pp} \frac{\partial}{\partial w_{pp}}; \quad (13)$$

the coefficients Ψ appearing in here is as follows: under the infinitesimal transformation described by X , the partial derivatives get transformed as $w_p \rightarrow w_p + \varepsilon \Psi_p$, $w_t \rightarrow w_t + \varepsilon \Psi_t$, $w_{pp} \rightarrow w_{pp} + \varepsilon \Psi_{pp}$.

When X is the generalized scaling (12), i.e. for the mapping $p \rightarrow (1 - \varepsilon)p$, $t \rightarrow (1 + \varepsilon\sigma)t$, $w \rightarrow (1 + \varepsilon(\nu p^2 w^\delta))w$, the corresponding Ψ can be easily computed by the so-called prolongation formula [8–10] to be

$$\begin{aligned} \Psi_t &= -\sigma w_t + \nu q p^2 w^\delta; \\ \Psi_p &= w_p + 2\nu p w^{(1+\delta)} + \nu(1 + \delta)p^2 w^\delta; \\ \Psi_{pp} &= 2w_{pp} + 2\nu w^{(1+\delta)} + \nu w^{(\delta-1)}(1 + \delta) [\delta p^2 w_p^2 + p w(4w_p + p w_{pp})]. \end{aligned} \quad (14)$$

These determine how the differential equation (4) of interest here is transformed under X .

In computational terms, this is done by applying Y on (4), and substituting for w_t according to (4) itself; the obtained expression must be zero for X to be a symmetry of the equation. Proceeding in this way, we obtain a condition which we write in compact form (see below for the explicit expressions of the functions A_k) as

$$A_0(p, t, w) + A_1(p, t, w) w_p + A_2(p, t, w) w_{pp} = 0. \quad (15)$$

The functions A_k must vanish separately for (15) to be satisfied.

The explicit expression of A_2 is

$$A_2 = -\gamma_0(2 + \sigma) - \frac{\gamma_1 p_c^2 (p_c^2(2 + \sigma) + (4 + \sigma)p^2)}{[p_c^2 + p^2]^2}. \quad (16)$$

The limit of this for large $|p|$ is nonzero unless we choose, as we do from now on,

$$\sigma = -2. \quad (17)$$

With this choice, X reads

$$X = -p\partial_p - 2t\partial_t + \nu p^2 w^q \partial_w \quad (18)$$

and A_2 reduces to

$$A_2 = \frac{-2\gamma_1 p_c^2 p^2}{(p_c^2 + p^2)^2}; \quad (19)$$

therefore, $A_2 \rightarrow 0$ for $|p| \rightarrow \infty$.

As for A_1 , with the choice (17) it reads

$$\begin{aligned} A_1 &= [2p((p_c^2 + p^2)(\alpha p_c^4 - 2(1 + \delta)\gamma_0 \nu w^\delta (p_c^2 + w^2)^2) \\ &\quad - 2\gamma_1 p_c^2((1 + \delta)\nu p_c^4 w^\delta + p^2(-1 + \nu w^\delta p^2 + \delta \nu w^\delta p^2) \\ &\quad + p_c^2(1 + 2\nu w^\delta p^2 + 2\delta \nu w^\delta p^2)))] \times [p_c^2 + p^2]^{-3}. \end{aligned} \quad (20)$$

This is a more involved expression, but it is still easy to check that in the limit $|p| \rightarrow \infty$ and $w \rightarrow 0$, recall (3), we get $A_1 \rightarrow 0$.

Finally, and again with (17), A_0 reads

$$\begin{aligned} A_0 &= [w(-2\nu w^\delta (p_c^2 + p^2)(\gamma_1 p_c^2 (p_c^2 - p^2) + \\ &\quad + \gamma_0(p_c^2 + p^2)^2) + \\ &\quad + \alpha p_c^2(-(2 + \delta)\nu w^\delta p^6) - 2p_c^2 p^2(3 + 2\nu w^\delta p^2) + \\ &\quad + p_c^4(2 - 2\nu w^\delta p^2 + \delta \nu w^\delta p^2))] \times \\ &\quad \times [p_c^2 + p^2]^{-3}. \end{aligned} \quad (21)$$

Once again, for $|p| \rightarrow \infty$ and $w \rightarrow 0$ we get $A_0 \rightarrow 0$.

This computation shows on the one hand that the one-parameter group of transformations generated by (18) is not a symmetry of the equation (4), as the A_k are not identically zero; on the other hand, it shows that (18) is an asymptotic symmetry for this equation, defined in the function space identified by (3). No vector field of the form (12) with $\sigma \neq -2$ is an asymptotic symmetry of (4); hence (18) is the only invariance generator for (5) which is also an asymptotic symmetry of (4).

Let us discuss the effect on (18) on the normalization condition (3). Using (8) we obtain that a generic function $w(p, t)$ is mapped by (18) into $\hat{w} = w + \varepsilon \delta w$ with $\delta w = \nu p^2 w^q + p w_p + 2t w_t$; on solutions to (1) this reads

$$\delta w = \nu p^2 w^q + p w_p - 2t \frac{\partial}{\partial p}(h w - g w_p). \quad (22)$$

With $I[w] := \int_{-\infty}^{+\infty} w(p, t) dp$, we obviously have $I[\hat{w}] = I[w] + \varepsilon I[\delta w]$. In view of (22), the latter amounts to

$$I[\delta w] = \nu I[p^2 w^q] + I[p w_p] + 2t I[\partial_p(h w - g w_p)]. \quad (23)$$

It is immediate to see, using also an integration by parts for the last one, that (3) implies the finiteness of the last two integrals in (23).

As for $I[p^2 w^q]$, note that (3) holds if, for $|p| \rightarrow \infty$, $w(p) \simeq 1/p^{2k}$ with $k > 1/2$. For general solutions $w(p, t)$, the condition of normalizable variation $kq > 3/2$ is more restrictive than the normalization condition $k > 1/2$ for all q in the physical range $q < 3$.

Note that for $w(p)$ given by (5), $k = 1/\delta$ and the condition $kq > 3/2$ holds for all $q < 3$; that is, functions which are near to the stationary solution $w_0(p)$ will always have normalizable variation under (18).

Thus, strictly speaking, our method should be applied only on functions satisfying $|I[p^2 w^q]| < \infty$, i.e. decaying (for $|p| \rightarrow \infty$) faster than $1/|p|^{3/q}$. As observed above, this includes all functions near to the Tsallis distribution (5), for all q in the physical range.

The (asymptotic) invariance of (4) under (18) could also be analyzed using the systematic procedure of [1]; this requires to introduce symmetry-adapted coordinates (v, y, σ) , with v the dependent variable. These are the coordinates in which (18) is simply $X = -2\sigma\partial_\sigma$, and its action on derivatives up to order two is described by $Y = X + 2v_\sigma(\partial/\partial v_\sigma)$ [8–10]. In the present case, adapted coordinates are $\sigma = t$, $y = p^2/t$, $v = w^{-\delta} - (\nu\delta/2)p^2$.

Finally, let us briefly mention differences with the work by Abe [7]. Abe considered a dilation symmetry of (4), based on an auxiliary field $\Lambda(p)$ which could be not normalizable. The Abe symmetry has generator $G = \int p\Lambda_p w dp$. In the language of symmetry theory [8–10] this is a nonlocal symmetry (it depends on an integral over p rather than just on the value of w at the point p), whereas here we considered local ones. Moreover Abe worked in canonical formalism, which requires to select a symplectic structure, whereas here we did not consider

any additional structure. The two approaches are thus quite different; it appears that symmetries considered by Abe are more general, while the one considered here has the advantage of being a *local* one.

Let us summarize our discussion. We have considered the equation for the marginal Wigner distribution $w(p, t)$ of the momentum p at time t in an optical lattice from the point of view of symmetry analysis. It is known that this equation admits the distribution $w_0(p)$ as a stationary solution.

We have identified the generalized scaling invariance group of $w_0(p)$, described by (11) and generated by $X = -p\partial_p + (2\beta Z^{q-1} p^2 w^q)\partial_w$.

We passed then to consider generalized scalings (12) acting on the time variable as well; it is easily seen that these cannot be symmetries of the equation (4). We investigated then if they may be *asymptotic symmetries* for that equation, and observed that a necessary condition for this is provided by (17). With this choice, it turns out that we have indeed an asymptotic symmetry, whose generator is (18), of our equation.

This shows that the Tsallis distribution (4) is an invariant function for a transformation X which is an asymptotic symmetry (for large $|p|$) of the equation under study, and hence that *the scale-free asymptotic behavior of $w_0(p)$ is a consequence of the asymptotic symmetry properties of the equation (4)* describing the characteristics of the optical lattice.

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